MATH 323: Algebraic Topology

Exam 2023

Thursday, June 29th

9:15 - 12:15

All your answers need to be justified and explained carefully, unless indicated otherwise. You are allowed to use all the results from the course and the exercises, but you need to indicate clearly, when you do so. It is indicated behind each question how many points it gives. There is a total of 100 points.

As always, S^n denotes the n-sphere

$$S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\},$$

and D^n the closed n-disk

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

Good luck!!!

1. For each of the following statements, state whether it is true (T) or false (F). No justification is needed. You obtain 3 points for each correct answer, 0 points for not answering a question, and -3 points for every incorrect answer. In total, you will not obtain less than 0 points for this question. [**30 points**]

   (a) The n-skeleton of a CW complex X is closed in X.

   (b) There exists an orientable connected compact manifold of dimension 2023 whose Euler characteristic is 2023.

   (c) Let X be a CW complex and $A \subset X$ a non-empty subcomplex. Then $H_k(X, A) \simeq \tilde{H}_k(X/A)$ for all $k \geq 0$.

   (d) Let M be an n-dimensional manifold such that $H_1(M) = 0$. Then M is orientable.

   (e) Let $h\colon S^2 \to S^5$ be an embedding. Then $\tilde{H}_i(S^5 \setminus h(S^2)) = 0$ for all $i \geq 0$ such that $i \neq 2$.

   (f) The degree of a surjective map $f\colon S^n \to S^n$ is always non-zero.

   (g) Let X be a topological space and $A \subset X$ a retract. Then the homomorphisms $H_k(A) \to H_k(X)$ induced by the inclusion are injective for all $k \geq 0$.

   (h) Let $n \geq 1$ and let $f\colon S^{2n} \to S^{2n}$ be a continuous map such that $f(x) = f(-x)$ for all $x \in S^{2n}$, then f is of degree 0.

   (i) Let X be a topological space and $A \subset X$, then $H_0(X, A) = 0$ if and only if A intersects each path component of X non-emptily.

   (j) Let X be a finite CW complex whose number of n-cells is exactly k. Then $H_k(X) \simeq \mathbb{Z}^k$.

2. Let $U, V \subset \mathbb{R}^n$ be two open subsets such that $U \cup V = \mathbb{R}^n$. Assume that $U \cap V$ has only finitely many path-components

   (a) Show that U and V have only finitely many path-components. [**5 points**]

   (b) Let α be the number of path-components of U, β the number of path-components of V, and γ the number of path-components of $U \cap V$. Show that $\gamma = \alpha + \beta - 1$. [**5 points**]

(c) Assume that $U \cap V$ is simply connected. What can we say about $H_1(U)$ and $H_1(V)$? Justify your answer. [**10 points**]

(d) Assume that two points $x, y \in U \cap V$ can be connected by a path in U and by a path in V. Can x and y also be connected by a path in $U \cap V$? Justify your answer. [**10 points**]

3. Let $\mathbb{C}P^n$ be the complex projective n-space and let $A \subset \mathbb{C}P^n$ be a subset consisting of 10 points. Compute the relative homology groups $H_k(\mathbb{C}P^n, A)$ for all $k \geq 0$. [**10 points**]

4. Let A be a finitely generated abelian group. Let A' be the quotient of A by its torsion part (i.e., the quotient of A by its subgroup of elements of finite order). Then $A' \simeq \mathbb{Z}^r$ for some r. A homomorphism $\varphi\colon A \to A$ induces a homomorphism $\varphi'\colon A' \to A'$. If we identify A' with $\mathbb{Z}^r$, then φ' is given by left-multiplication with an integer matrix M. We define the *trace* of φ to be the trace of M. Since the trace is invariant under conjugation, the trace of f does not depend on the choice of isomorphism between A' and $\mathbb{Z}^r$.

Let X be a finite CW complex and $f\colon X \to X$ a continuous self-map. For each $i \geq 0$, the map f induces a homomorphism $f_*\colon H_i(X) \to H_i(X)$. Let t_i be the trace of this homomorphism. We define the *Lefschetz number* $\Delta(f)$ of f as

$$\Delta(f) = \sum_{i \geq 0} (-1)^i t_i.$$

We say that a finite CW complex X has the *Lefschetz property* if the following holds: If a continuous map $f\colon X \to X$ has no fixed points, then $\Delta(f) = 0$.

The *Lefschetz fixed point theorem* states that every non-empty finite Δ-complex has the Lefschetz property.

Answer the following questions by using these definitions and observations (you do not need to reprove the facts explained in this introductory paragraph).

(a) Show that the Lefschetz number of the identity map is equal to the Euler characteristic of X. [**10 points**]

(b) Use the Lefschetz fixed point theorem to show that if X is a contractible finite Δ-complex and $f\colon X \to X$ a continuous map, then f has a fixed point. [**5 points**]

(c) Use the Lefschetz fixed point theorem to show that every continuous map $f\colon \mathbb{R}P^2 \to \mathbb{R}P^2$ has a fixed point, where $\mathbb{R}P^2$ denotes the real projective plane. [**5 points**]

5. (a) Construct a topological space X satisfying the following conditions (justify your answer):

   - $H_0(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$,
   - $H_1(X) = 0$,
   - $H_2(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

   [**5 points**]

   (b) What are the cohomology groups $H^k(X; \mathbb{Z})$? (justify your answer) [**5 points**]